WHAT ARE THE SHAPES OF EMBEDDED MINIMAL SURFACES AND WHY?

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Summary. Minimal surfaces with uniform curvature (or area) bounds have been well understood and the regularity theory is complete, yet essentially nothing was known without such bounds. We discuss here the theory of embedded (i.e., without self-intersections) minimal surfaces in Euclidean space $\mathbf{R}^3$ without a priori bounds. The study is divided into three cases, depending on the topology of the surface. Case one is where the surface is a disk, in case two the surface is a planar domain (genus zero), and the third case is that of finite (non-zero) genus. The complete understanding of the disk case is applied in both cases two and three.

As we will see, the helicoid, which is a double spiral staircase, is the most important example of an embedded minimal disk. In fact, we will see that every such disk is either a graph of a function or part of a double spiral staircase. The helicoid was discovered to be a minimal surface by Meusnier in 1776.

For planar domains the fundamental examples are the catenoid, also discovered by Meusnier in 1776, and the Riemann examples discovered by Riemann in the beginning of the 1860s[1]. Finally, for general fixed genus an important example is the recent example by Hoffman-Weber-Wolf of a genus one helicoid.

In the last section we discuss why embedded minimal surfaces are automatically proper. This was known as the Calabi-Yau conjectures for embedded surfaces. For immersed surfaces there are counter-examples by Jorge-Xavier and Nadirashvili.

0.1. Shape of things that are in equilibrium.

What are the possible shapes of natural objects in equilibrium and why?

When a closed wire or a frame is dipped into a soap solution and is raised up from the solution, the surface spanning the wire is a soap film. The soap film is in a state of equilibrium. What are the possible shapes of soap films and why? Or why is DNA like a double spiral staircase? "What..?" and "why..?" are fundamental questions which, when answered, help us understand the world we live in. The answer to any question about the shape of natural objects is bound to involve mathematics.

Soap films, soap bubbles, and surface tension were extensively studied by the Belgian physicist and inventor (of the stroboscope) Plateau in the first half of the nineteenth century. At least since his studies, it has been known that the right mathematical model for a soap

[1]Riemann worked on minimal surfaces in the period 1860-1861. He died in 1866. The Riemann example was published post-mortem in 1867 in an article edited by Poggendorf.

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film is a minimal surface[2] – the soap film is in a state of minimum energy when it is covering the least possible amount of area.

We will discuss here the answer to the question: "What are the possible shapes of embedded minimal surfaces in $\mathbf{R}^3$ and why?"

0.2. **Critical points, minimal surfaces.** Let $\Sigma \subset \mathbf{R}^3$ be a smooth orientable surface (possibly with boundary) with unit normal $\mathbf{n}_\Sigma$. Given a function ϕ in the space $C_0^\infty(\Sigma)$ of infinitely differentiable (i.e., smooth), compactly supported functions on Σ, consider the one-parameter variation

$$\Sigma_{t,\phi} = \{x + t\,\phi(x)\,\mathbf{n}_\Sigma(x) | x \in \Sigma\}\,. \tag{0.1}$$

The so-called first variation formula of area is the equation (integration is with respect to the area of Σ)

$$\left.\frac{d}{dt}\right|_{t=0} \text{Area}(\Sigma_{t,\phi}) = \int_\Sigma \phi\, H\,, \tag{0.2}$$

where H is the mean curvature of Σ and the mean curvature is the sum of the principal curvatures κ_1, κ_2. (When Σ is non-compact, $\Sigma_{t,\phi}$ in (0.2) is replaced by $\Gamma_{t,\phi}$, where Γ is any compact set containing the support of ϕ.) The surface Σ is said to be a *minimal* surface (or just minimal) if

$$\left.\frac{d}{dt}\right|_{t=0} \text{Area}(\Sigma_{t,\phi}) = 0 \qquad \text{for all } \phi \in C_0^\infty(\Sigma) \tag{0.3}$$

or, equivalently by (0.2), if the mean curvature H is identically zero. Thus Σ is minimal if and only if it is a critical point for the area functional. Moreover, when Σ is minimal, $\kappa_1 = -\kappa_2$ (since $H = \kappa_1 + \kappa_2 = 0$) and the Gaussian curvature $\mathrm{K}_\Sigma = \kappa_1\,\kappa_2$ is non-positive.

0.3. **Minimizers and stable minimal surfaces.** Since a critical point is not necessarily a minimum the term "minimal" is misleading, but it is time-honored. The equation for a critical point is also sometimes called the Euler-Lagrange equation. A computation shows that if Σ is minimal, then

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \text{Area}(\Sigma_{t,\phi}) = -\int_\Sigma \phi\, L_\Sigma \phi\,, \qquad \text{where } L_\Sigma \phi = \Delta_\Sigma \phi + |A|^2 \phi \tag{0.4}$$

is the second variational (or Jacobi) operator. Here Δ_Σ is the Laplacian on Σ and A is the second fundamental form of Σ. So A is the covariant derivative of the unit normal of Σ and $|A|^2 = \kappa_1^2 + \kappa_2^2 = -2\,\kappa_1\,\kappa_2 = -2\mathrm{K}_\Sigma$, where κ_1, κ_2 are the principal curvatures (recall that since Σ is minimal $\kappa_1 = -\kappa_2$). A minimal surface Σ is said to be stable if

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \text{Area}(\Sigma_{t,\phi}) \geq 0 \qquad \text{for all } \phi \in C_0^\infty(\Sigma)\,. \tag{0.5}$$

One can show that a minimal graph is stable and, more generally, so is a multi-valued minimal graph (see below for the precise definition).

Throughout, let x_1, x_2, x_3 be the standard coordinates on $\mathbf{R}^3$. For $y \in \Sigma \subset \mathbf{R}^3$ and $s > 0$, the extrinsic balls are $B_s(y) = \{x \in \mathbf{R}^3 | |x - y| < s\}$.

[2]The field of minimal surfaces dates back to the publication in 1762 of Lagrange's famous memoir "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies". Euler had already, in a paper published in 1744, discussed minimizing properties of the surface now known as the catenoid, but he only considered variations within a certain class of surfaces.

0.4. **Embedded = without self-intersections.** Our surfaces will all be without self-intersections, i.e., they will be embedded. By embedded we mean a smooth injective immersion from an abstract surface into $\mathbf{R}^3$.

0.5. **Topology of surfaces.** The classification of minimal surfaces will essentially only depend on the topology of the surface and on whether or not the surface has a point where the curvature is large.

Compact orientable surfaces without boundaries are classified by their genus, a nonnegative integer. Genus $= 0$ corresponds to a sphere, genus $= 1$ to the torus, a model of which is the surface of an object formed by attaching a "suitcase handle" to a sphere. A surface of genus $= k$ is modelled by the surface of a sphere to which k-handles have been attached. A compact orientable surface with boundary is one formed by taking one of these surfaces and removing a number of disjoint disks. The genus of the surface with boundary is the genus of the original object, and the boundary corresponds to the edges of the surface created by disk removal. In particular, a surface with genus 0 and non-empty boundary is a planar domain, i.e., it can be obtained from the disk in the plane by removing a number of disjoint sub-disks. This is because it can be obtained from the sphere by removing a number of disks and after removing the first disk from the sphere, we have a disk in a plane. Sometimes we will talk about surfaces that are simply connected. By this we will mean that every loop on the surface can be shrunk (without leaving the surface) to a point curve. One can easily see that the only simply connected surfaces are the disk and the sphere.

1. Disks

There are two local models for embedded minimal disks. One model is the plane (or, more generally, a minimal graph) and the other is a piece of a helicoid.

1.1. **Minimal graphs and the helicoid.** The derivation of the equation for a minimal graph goes back to Lagrange's 1762 memoir. (Note that if Ω is a simply connected domain in $\mathbf{R}^2$ and u is a real valued function, the graph of u, i.e., the set $\{(x_1, x_2, u(x_1, x_2)) \,|\, (x_1, x_2) \in \Omega\}$, is a disk.) This gives a large class of embedded minimal disks where the function is defined over a proper subset of $\mathbf{R}^2$, however by a classical theorem of Bernstein from 1916 entire (i.e., where $\Omega = \mathbf{R}^2$) minimal graphs are planes.

The second model comes from the helicoid which was discovered by Meusnier in 1776[3]. The helicoid is a "double spiral staircase" given by sweeping out a horizontal line rotating at a constant rate as it moves up a vertical axis at a constant rate. Each half-line traces out a spiral staircase and together the two half-lines trace out (up to scaling) the double spiral staircase

$$(s \cos t, s \sin t, t)\,, \qquad \text{where } s,\, t \in \mathbf{R}\,. \tag{1.1}$$

Anyone who has climbed a spiral staircase knows that the stairs become steep in the center. The tangent plane to the helicoid at a point on the vertical axis is a vertical plane; thus the helicoid is not a graph over the horizontal plane. In fact, as we saw earlier any minimal surface has non-positive curvature, for the helicoid the curvature is most negative

[3]Meusnier had been a student of Monge. He also discovered that the surface now known as the catenoid is minimal in the sense of Lagrange, and he was the first to characterize a minimal surface as a surface with vanishing mean curvature. Unlike the helicoid, the catenoid is not topologically a plane but rather a cylinder. (The catenoid will be explained later; see (1.5).)

along the axis and converges asymptotically to zero as one moves away from the axis. This corresponds to that as one moves away from the axis larger and larger pieces of the helicoid are graphs.

For the our results about embedded minimal disks (see Sub-section 1.3 below) it will be important to understand a sequence of helicoids obtained from a single helicoid by rescaling as follows:

> Consider the sequence $\Sigma_i = a_i\,\Sigma$ of rescaled helicoids where $a_i \to 0$. (That is, rescale $\mathbf{R}^3$ by a_i, so points that used to be distance d apart will in the rescaled $\mathbf{R}^3$ be distance $a_i\,d$ apart.) The curvatures of this sequence of rescaled helicoids are blowing up (i.e., the curvatures go to infinity) along the vertical axis. The sequence converges (away from the vertical axis) to a foliation by flat parallel planes. The singular set (the axis) then consists of removable singularities.

1.2. **Multi-valued graphs, spiral staircases, double spiral staircases.** To be able to give a precise meaning to the statement that the helicoid is a double spiral staircase we will need the notion of a multi-valued graph, each staircase will be a multi-valued graph. Intuitively, a multi-valued graph is a surface covering an annulus, such that over a neighborhood of each point of the annulus, the surface consists of N graphs. To make this notion precise, let D_r be the disk in the plane centered at the origin and of radius r and let $\mathcal{P}$ be the universal cover of the punctured plane $\mathbf{C} \setminus \{0\}$ with global polar coordinates (ρ, θ) so $\rho > 0$ and $\theta \in \mathbf{R}$. An *N-valued graph* on the annulus $D_s \setminus D_r$ is a single valued graph of a function u over $\{(\rho,\theta) \,|\, r < \rho \leq s\,,\, |\theta| \leq N\,\pi\}$. For working purposes, we generally think of the intuitive picture of a multi-sheeted surface in $\mathbf{R}^3$, and we identify the single-valued graph over the universal cover with its multi-valued image in $\mathbf{R}^3$.

The multi-valued graphs that we will consider will all be embedded, which corresponds to a non-vanishing separation between the sheets (or the floors). If Σ is the helicoid, then $\Sigma \setminus \{x_3 - \text{axis}\} = \Sigma_1 \cup \Sigma_2$, where Σ_1, Σ_2 are ∞-valued graphs on $\mathbf{C} \setminus \{0\}$. Σ_1 is the graph of the function $u_1(\rho,\theta) = \theta$ and Σ_2 is the graph of the function $u_2(\rho,\theta) = \theta + \pi$. ($\Sigma_1$ is the subset where $s > 0$ in (1.1) and Σ_2 the subset where $s < 0$.) In either case the separation between the sheets is constant, equal to $2\,\pi$. A *multi-valued minimal graph* is a multi-valued graph of a function u satisfying the minimal surface equation.

1.3. **Structure of embedded minimal disks.** All of our results, for disks as well as for other topological types, require only a piece of a minimal surface. In particular, the surfaces may well have boundaries. This is a major point and makes the results particularly useful.

The following is the main structure theorem for embedded minimal disks[4]:

Theorem 1.2. Any embedded minimal disk in $\mathbf{R}^3$ is <u>either</u> a graph of a function <u>or</u> part of a double spiral staircase. In particular, if for some point the curvature is sufficiently large, then the surface is part of a double spiral staircase (it can be approximated by a piece of a rescaled helicoid). On the other hand, if the curvature is below a certain threshold everywhere, then the surface is a graph of a function.

As a consequence of this structure theorem we get the following compactness result:

[4]See [CM1]–[CM4] for the precise statements, as well as proofs, of the results of this section.

Corollary 1.3. A sequence of embedded minimal disks with curvatures blowing up (i.e., going to infinity[5]) at a point mimics the behavior of a sequence of rescaled helicoids with curvature going to infinity; see the discussion of rescaled helicoids at the end of Sub-section 1.1.

1.4. **A consequence for sequences that are ULSC.** Sequences of planar domains which are not simply connected are, after passing to a subsequence, naturally divided into two separate cases depending on whether or not the topology is concentrating at points. To distinguish between these cases, we will say that a sequence of surfaces $\Sigma_i^2 \subset \mathbf{R}^3$ is *uniformly locally simply connected* (or ULSC) if for each $x \in \mathbf{R}^3$, there exists a constant $r_0 > 0$ (depending on x) so that for all $r \leq r_0$, and every surface Σ_i

$$\text{each connected component of } B_r(x) \cap \Sigma_i \text{ is a disk.} \tag{1.4}$$

For instance, a sequence of rescaled catenoids where the necks shrink to zero is not ULSC, whereas a sequence of rescaled helicoids is.

The catenoid is the minimal surface in $\mathbf{R}^3$ given by rotating the curve $s \to (\cosh s, s)$ around the x_3-axis, i.e., the surface

$$(\cosh s\, \cos t, \cosh s\, \sin t, s) \text{ where } s,\, t \in \mathbf{R}\,. \tag{1.5}$$

Applying the above structure theorem for disks to ULSC sequences gives that there are only two <u>local models</u> for such surfaces. That is, locally in a ball in $\mathbf{R}^3$, one of following holds:

- The curvatures are bounded and the surfaces are locally <u>graphs</u> over a plane.
- The curvatures blow up and the surfaces are locally <u>double spiral staircases</u>.

Both of these cases are illustrated by taking a sequence of rescalings of the helicoid; the first case occurs away from the axis, while the second case occurs on the axis. If we take a sequence $\Sigma_i = a_i\, \Sigma$ of rescaled helicoids where $a_i \to 0$, then the curvature blows up along the vertical axis but is bounded away from this axis. Thus, we get that

- The intersection of the rescaled helicoids with a ball <u>away from</u> the vertical axis gives a collection of graphs over the plane $\{x_3 = 0\}$.
- The intersection of the rescaled helicoids with a ball <u>centered on</u> the vertical axis gives a double spiral staircase.

1.5. **Two key ideas behind the proof of the structure theorem for disks.** The first of these key ideas says that if the curvature of such a disk Σ is large at some point $x \in \Sigma$, then near x a multi-valued graph forms (in Σ), and this extends (in Σ) almost all the way to the boundary of Σ[6]. Moreover, the inner radius, r_x, of the annulus where the multi-valued graph is defined is inversely proportional to $|A|(x)$, and the initial separation between the sheets is bounded by a constant times the inner radius.

An important ingredient in the proof of Theorem 1.2 is that general embedded minimal disks with large curvature at some interior point can be built out of N-valued graphs. In other words, any embedded minimal disk can be divided into pieces each of which is an

[5]Recall that for a minimal surface in $\mathbf{R}^3$ the curvature $K = -\frac{1}{2}|A|^2$ is non-positive; so by that the curvatures of a sequence is going to infinity we mean that $K \to -\infty$ or equivalently $|A|^2 \to \infty$.

[6]Recall that our results require only that we have a piece of a minimal surface and thus it may have boundary.

N-valued graph. Thus the disk itself should be thought of as being obtained by stacking these pieces (graphs) on top of each other[7].

The second key result (Theorem 1.6) is a curvature estimate for embedded minimal disks in a half-space. As a corollary (Corollary 1.8 below) of this theorem, we get that the set of points in an embedded minimal disk where the curvature is large lies within a cone, and thus the multi-valued graphs, whose existence was discussed above, will all start off within this cone.

The curvature estimate for disks in a half-space is the following:

Theorem 1.6. There exists $\epsilon > 0$ such that for all $r_0 > 0$, if $\Sigma \subset B_{2r_0} \cap \{x_3 > 0\} \subset \mathbf{R}^3$ is an embedded minimal disk with $\partial\Sigma \subset \partial B_{2r_0}$, then for all components Σ' of $B_{r_0} \cap \Sigma$ which intersect $B_{\epsilon r_0}$

$$\sup_{x \in \Sigma'} |A_\Sigma(x)|^2 \leq r_0^{-2} \, . \tag{1.7}$$

Theorem 1.6 is an interior estimate where the curvature bound, (1.7), is on the ball B_{r_0} of one half of the radius of the ball B_{2r_0} containing Σ. This is just like a gradient estimate for a harmonic function where the gradient bound is on one half of the ball where the function is defined.

Using the minimal surface equation and the fact that Σ' has points close to a plane, it is not hard to see that, for $\epsilon > 0$ sufficiently small, (1.7) is equivalent to the statement that Σ' is a graph over the plane $\{x_3 = 0\}$.

We will often refer to Theorem 1.6 as *the one-sided curvature estimate* (since Σ is assumed to lie on one side of a plane). Note that the assumption in Theorem 1.6 that Σ is simply connected (i.e., that Σ is a disk) is crucial, as can be seen from the example of a rescaled catenoid. Rescaled catenoids converge (with multiplicity two) to the flat plane. Likewise, by considering the universal cover of the catenoid, one sees that Theorem 1.6 requires the disk to be embedded, and not just immersed.

In the proof of Theorem 1.2, the following (direct) consequence of Theorem 1.6 (with the 2-valued graph playing the role of the plane $\{x_3 = 0\}$) is needed.

Corollary 1.8. If an embedded minimal disk contains a 2-valued graph over an annulus in a plane, then away from a cone with axis orthogonal to the 2-valued graph the disk consists of multi-valued graphs over annuli in the same plane.

By definition, if $\delta > 0$, then the (convex) cone with vertex at the origin, cone angle $(\pi/2 - \arctan \delta)$, and axis parallel to the x_3-axis is the set

$$\{x \in \mathbf{R}^3 \mid x_3^2 \geq \delta^2 \, (x_1^2 + x_2^2)\} \, . \tag{1.9}$$

1.6. **Uniqueness theorems.** Using the above structure theorem for disks Meeks-Rosenberg, [MeRo], proved that the plane and the helicoid are the only complete properly embedded simply-connected minimal surfaces in $\mathbf{R}^3$ (the assumption of properness can in fact be removed by [CM6]; see Section 4). Catalan had proved in 1842 that any complete ruled

[7]The parallel to the helicoid is striking. Half of the helicoid, i.e., $(s \cos t, s \sin t, t)$, where $s > 0$ and $t \in \mathbf{R}$, can be obtained by stacking the N-valued graphs, $(s \cos(k \, N \, 2\pi + t), s \sin(k \, N \, 2\pi + t), k \, N \, 2\pi + t)$, where $s > 0$, $N \, 2\pi > t \geq 0$, and k is an integer, on the top of each other.

minimal surface is either a plane or a helicoid. A surface is said to be *ruled* if it has the parametrization

$$X(s,t) = \beta(t) + s\,\delta(t), \qquad \text{where } s,\, t \in \mathbf{R}, \tag{1.10}$$

and β and δ are curves in $\mathbf{R}^3$. The curve $\beta(t)$ is called the *directrix* of the surface, and a line having $\delta(t)$ as direction vector is called a *ruling*. For the helicoid in (1.1), the x_3-axis is a directrix, and for each fixed t the line $s \to (s\,\cos t, s\,\sin t, t)$ is a ruling.

For cylinders there is a corresponding uniqueness theorem. Namely, combining [S], [Cn] (see also [CM7]), and [CM6] one has that any complete embedded minimal cylinder in $\mathbf{R}^3$ is a catenoid.

Conjecturally similar uniqueness theorems should hold for other planar domains and surfaces of fixed (non-zero) genus; cf. [MeP], [HWW].

2. Planar domains

We describe next two main structure theorems for *non-simply connected* embedded minimal planar domains. (The precise statements of these results and their proofs can be found in [CM5].)

The first of these asserts that any such surface without small necks[8] can be obtained by gluing together two oppositely-oriented double spiral staircases. Note that when one glues two oppositely oriented double spiral staircases together, then one remains at the same level if one circles both axes.

The second gives a "pair of pants" decomposition of any such surface when there are small necks, cutting the surface along a collection of short curves. After the cutting, we are left with graphical pieces that are defined over a disk with either one or two sub-disks removed (a topological disk with two sub-disks removed is called a "pair of pants").

Both of these structures occur as different extremes in the two-parameter family of minimal surfaces known as the Riemann examples.

2.1. **The catenoid and the Riemann examples.**

When the sequence is no longer ULSC, then there are other local models for the surfaces. The simplest example is a sequence of rescaled catenoids.

A sequence of rescaled catenoids converges with multiplicity two to the flat plane. The convergence is in the C^∞ topology except at 0 where $|A|^2 \to \infty$. This sequence of rescaled catenoids is not ULSC because the simple closed geodesic on the catenoid – i.e., the unit circle in the $\{x_3 = 0\}$ plane – is non-contractible and the rescalings shrink it down to the origin.

One can get other types of curvature blow-up by considering the family of embedded minimal planar domains known as the Riemann examples. Modulo translations and rotations, this is a two-parameter family of periodic minimal surfaces, where the parameters can be thought of as the size of the necks and the angle from one fundamental domain to the next. By choosing the two parameters appropriately, one can produce sequences of Riemann examples that illustrate both of the two structure theorems:

[8]By "without small necks" we mean that the intersection of the surface with all extrinsic balls with sufficiently small radii consists of simply connected components; cf. the notion of ULSC for sequences above.

(1) If we take a sequence of Riemann examples where the neck size is fixed and the angles go to $\frac{\pi}{2}$, then the surfaces with angle near $\frac{\pi}{2}$ can be obtained by gluing together two oppositely-oriented double spiral staircases. Each double spiral staircase looks like a helicoid. This sequence of Riemann examples converges to a foliation by parallel planes. The convergence is smooth away from the axes of the two helicoids (these two axes are the singular set where the curvature blows up). The sequence is ULSC since the size of the necks is fixed and thus illustrates the first structure theorem, Corollary 2.2 below.

(2) If we take a sequence of examples where the neck sizes go to zero, then we get a sequence that is *not* ULSC. However, the surfaces can be cut along short curves into collections of graphical pairs of pants. The short curves converge to points and the graphical pieces converge to flat planes except at these points, illustrating the second structure theorem, Corollary 2.4 below.

2.2. **Structure of embedded planar domains.** We describe next (Theorems 2.1 and 2.3 below) the two main structure theorems for *non-simply connected* embedded minimal planar domains. Each of these theorems has a compactness theorem as a consequence.

The first structure theorem deals with surfaces without small necks:

Theorem 2.1. Any non-simply connected embedded minimal planar domain without small necks can be obtained from gluing together two oppositely oriented double spiral staircases. Moreover, if for some point the curvature is large, then the separation between the sheets of the double spiral staircases is small. Note that since the two double spiral staircases are oppositely oriented, then one remains at the same level if one circles both axes.

The following compactness result is a consequence:

Corollary 2.2. A ULSC (but not simply connected) sequence of embedded minimal surfaces with curvatures blowing up has a subsequence that converges smoothly to a foliation by parallel planes away from two lines. The two lines are disjoint and orthogonal to the leaves of the foliation, and the two lines are precisely the points where the curvature is blowing up.

This is similar to the case of disks, except that we get two singular curves for non-disks as opposed to just one singular curve for disks.

Moreover, locally around each of the two lines the surfaces look like a helicoid around the axis and the orientation around the two axes are opposite.

Despite the similarity of Corollary 2.2 to the case of disks, it is worth noting that the results for disks do not alone give this result. Namely, even though the ULSC sequence consists locally of disks, the compactness result for disks was in the global case where the radii go to infinity. One might wrongly assume that Corollary 2.2 could be proven using the results for disks and a blow-up argument. However, one can construct local examples that show the difficulty of such an argument.

The second structure theorem deals with surfaces with small necks and gives a "pair of pants" decomposition:

Theorem 2.3. Any non-simply connected embedded minimal planar domain with a small neck can be cut along a collection of short curves. After the cutting, we are left with graphical pieces that are defined over a disk with either one or two sub-disks removed (a topological disk with two sub-disks removed is called a pair of pants).

Moreover, if for some point the curvature is large, then all the necks are very small.

The following compactness result is a consequence:

Corollary 2.4. A sequence of embedded minimal planar domains that are not ULSC, but with curvatures blowing up, has a subsequence that converges to a lamination by flat parallel planes.

3. Finite genus

3.1. **The genus one helicoid; structure results for general finite genus.** In a very recent paper Hoffman-Weber-Wolf, [HWW], have constructed a new complete embedded minimal surface in $\mathbf{R}^3$. They have shown that there exists a properly embedded minimal surface of genus one with a single end asymptotic to the end of the helicoid. We will refer to this minimal surface Σ as the genus one helicoid. Under scalings the sequence of genus one surfaces $a_i\,\Sigma$ where $a_i \to 0$ converges to the foliation of flat parallel planes in $\mathbf{R}^3$ just like a sequence of rescaled helicoids with curvatures blowing up. This is in fact a consequence of a general result that the theorems in the previous section, stated for planar domains, holds also for sequences with fixed genus with minor changes; see [CM5].

4. Embedded minimal surfaces are automatically proper

Implicit in all of the results mentioned above was an assumption that the minimal surfaces were proper. However, as we will see next, it turns out that embedded minimal surfaces are, in fact, automatically proper. This was the content of the Calabi-Yau conjectures which were proven to be true for embedded surfaces in [CM6].

4.1. **What is proper?** An immersed surface in $\mathbf{R}^3$ is *proper* if the pre-image of any compact subset of $\mathbf{R}^3$ is compact in the surface. For instance, a line is proper whereas a curve that spiral infinitely into a circle is not.

4.2. **The Calabi-Yau conjectures; the statements and examples.** The Calabi-Yau conjectures about surfaces date back to the 1960s. Their original form was given in 1965 where Calabi made the following two conjectures about minimal surfaces[9]:

Conjecture 4.1. "Prove that a complete minimal surface in $\mathbf{R}^3$ must be unbounded."

Calabi continued: "It is known that there are no compact minimal surfaces in $\mathbf{R}^3$ (or of any simply connected complete Riemannian 3-dimensional manifold with sectional curvature ≤ 0). A more ambitious conjecture is":

Conjecture 4.2. "A complete [non-flat] minimal surface in $\mathbf{R}^3$ has an unbounded projection in every line."

The <ins>immersed</ins> versions of these conjectures turned out to be false. Namely, Jorge and Xavier, [JX], constructed non-flat minimal immersions contained between two parallel planes in 1980, giving a counter-example to the immersed version of the more ambitious Conjecture 4.2. Another significant development came in 1996, when Nadirashvili, [N], constructed a

[9]S.S. Chern also promoted these conjectures at roughly the same time and they were revisited several times by S.T. Yau.

complete immersion of a minimal disk into the unit ball in $\mathbf{R}^3$, showing that Conjecture 4.1 also failed for immersed surfaces; see [LMaMo], for other topological types.

The main result in [CM6] is an effective version of properness for disks, giving a chord-arc bound[10]. Obviously, intrinsic distances are larger than extrinsic distances, so the significance of a chord-arc bound is the reverse inequality, i.e., a bound on intrinsic distances from above by extrinsic distances.

Given such a chord-arc bound, one has that as intrinsic distances go to infinity, so do extrinsic distances. Thus as an immediate consequence:

Theorem 4.3. A complete embedded minimal disk in $\mathbf{R}^3$ must be proper.

Theorem 4.3 gives immediately that the first of Calabi's conjectures is true for <u>embedded</u> minimal disks. In particular, Nadirashvili's examples cannot be embedded.

Another immediate consequence of the chord-arc bound together with the one-sided curvature estimate (i.e., Theorem 1.6) is a version of that estimate for intrinsic balls.

As a corollary of this intrinsic one-sided curvature estimate we get that the second, and more ambitious, of Calabi's conjectures is also true for <u>embedded</u> minimal disks. In particular, Jorge-Xavier's examples cannot be embedded. The second Calabi conjecture (for embedded disks) is an immediate consequence of the following half-space theorem:

Theorem 4.4. The plane is the only complete embedded minimal disk in $\mathbf{R}^3$ in a half-space.

The results for disks imply both of Calabi's conjectures and properness also for embedded surfaces with finite topology. A surface Σ is said to have finite topology if it is homeomorphic to a closed Riemann surface with a finite set of points removed or "punctures". Each puncture corresponds to an end of Σ.

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[10] A chord-arc bound is a bound above and below for the ratio of intrinsic to extrinsic distances.

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